where b_{ij} are the components of the Reynolds-stress anisotropy tensor **b**, referred to an inertial frame. Part of the assumption embodied in Eq. (11.221) is, therefore, that the principal axes of **b** do not rotate (relative to an inertial frame, following the mean flow). For some flows, for example those with significant mean streamline curvature, Girimaji (1997) argues that a better assumption is that the components of **b** are fixed relative to a particular rotating frame.

Exercise 11.30 Consider the algebraic stress model based on LRR-IP applied to a simple shear flow (Eqs. 11.130–11.134), and obtain expressions for b_{ij} in the limit $\mathcal{P}/\varepsilon \to \infty$. Are these values realizable? Are they consistent with RDT?

Exercise 11.31 Manipulate Eq. (11.219) to obtain

$$C_\mu = \frac{4b_{12}^2}{\mathcal{P}/\varepsilon}. \tag{11.222}$$

Hence use Eq. (11.134) to verify the expression for C_μ, Eq. (11.220).

Exercise 11.32 Consider a general model for the pressure–rate-of-strain that is linear in b_{ij} and in the mean velocity gradients, i.e., Eq. (11.135) with $f^{(2)} = f^{(6-8)} = 0$. Show that the corresponding algebraic stress model (Eq. 11.217) can be written

$$\begin{aligned} b_{ij} &= -g(\gamma_1 \widehat{S}_{ij} + \gamma_2 \left[\widehat{S}_{ik} b_{kj} + b_{ik}\widehat{S}_{kj} - \tfrac{2}{3}\widehat{S}_{k\ell} b_{\ell k}\delta_{ij}\right] \\ &\quad + \gamma_3[\widehat{\Omega}_{ik} b_{kj} - b_{ik}\widehat{\Omega}_{kj}]), \end{aligned} \tag{11.223}$$

where the coefficients (in general, and for the LRR models) are given in Table 11.5.

11.9.2 Nonlinear Turbulent Viscosity

The algebraic stress model equation Eq. (11.217) is an *implicit* equation for $\langle u_i u_j \rangle / k$, or equivalently for the anisotropy b_{ij}. Clearly, there is benefit in obtaining an *explicit* relation of the form

$$b_{ij} = \mathcal{B}_{ij}(\widehat{\mathbf{S}}, \widehat{\mathbf{\Omega}}), \tag{11.224}$$

where $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{\Omega}}$ are the normalized mean rate-of-strain and rotation tensors (Eqs. 11.136 and 11.137).

Table 11.5: Coefficients in Eq. (11.223) for different models.

Coefficient	General	LRR-IP	LRR-QI
g^{-1}	$-\frac{1}{2}f^{(1)} - 1 + \frac{\mathcal{P}}{\varepsilon}$	$C_R - 1 + \frac{\mathcal{P}}{\varepsilon}$	$C_R - 1 + \frac{\mathcal{P}}{\varepsilon}$
γ_1	$\frac{2}{3} - \frac{1}{2}f^{(3)}$	$\frac{2}{3}(1 - C_2)$	$\frac{4}{15}$
γ_2	$1 - \frac{1}{2}f^{(4)}$	$1 - C_2$	$\frac{1}{11}(5 - 9C_2)$
γ_3	$1 - \frac{1}{2}f^{(5)}$	$1 - C_2$	$\frac{1}{11}(1 + 7C_2)$

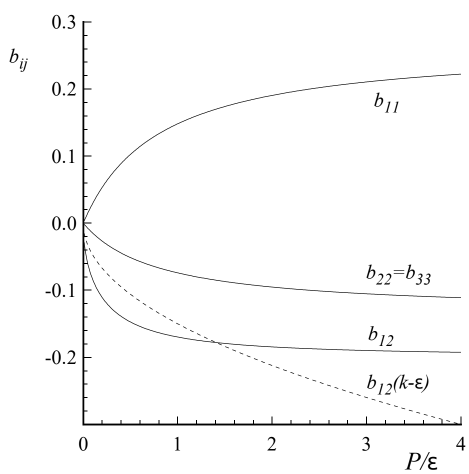

Figure 11.21: Reynolds-stress anisotropies as functions of $\mathcal{P}/\varepsilon$ according to the LRR-IP algebraic stress model. The dashed line shows b_{12} according to the k-ε model.

Table 11.6: Complete set of independent, symmetric, deviatoric functions $\widehat{\boldsymbol{\mathcal{T}}}^{(n)}$ of a deviatoric symmetric tensor $\widehat{\mathbf{S}}$ and an antisymmetric tensor $\widehat{\boldsymbol{\Omega}}$. Shown in matrix notation: braces denote traces, e.g., $\{\widehat{\mathbf{S}}^2\} = \widehat{S}_{ij}\widehat{S}_{ji}$.

$$
\begin{aligned}
\widehat{\boldsymbol{\mathcal{T}}}^{(1)} &= \widehat{\mathbf{S}}, & \widehat{\boldsymbol{\mathcal{T}}}^{(6)} &= \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}} + \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2 - \tfrac{2}{3}\{\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2\}\mathbf{I}, \\
\widehat{\boldsymbol{\mathcal{T}}}^{(2)} &= \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}} - \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}, & \widehat{\boldsymbol{\mathcal{T}}}^{(7)} &= \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}^2 - \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}, \\
\widehat{\boldsymbol{\mathcal{T}}}^{(3)} &= \widehat{\mathbf{S}}^2 - \tfrac{1}{3}\{\widehat{\mathbf{S}}^2\}\mathbf{I}, & \widehat{\boldsymbol{\mathcal{T}}}^{(8)} &= \widehat{\mathbf{S}}\widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2 - \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}, \\
\widehat{\boldsymbol{\mathcal{T}}}^{(4)} &= \widehat{\boldsymbol{\Omega}}^2 - \tfrac{1}{3}\{\widehat{\boldsymbol{\Omega}}^2\}\mathbf{I}, & \widehat{\boldsymbol{\mathcal{T}}}^{(9)} &= \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}^2 + \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2 - \tfrac{2}{3}\{\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2\}\mathbf{I}, \\
\widehat{\boldsymbol{\mathcal{T}}}^{(5)} &= \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2 - \widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}, & \widehat{\boldsymbol{\mathcal{T}}}^{(10)} &= \widehat{\boldsymbol{\Omega}}\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}^2 - \widehat{\boldsymbol{\Omega}}^2\widehat{\mathbf{S}}^2\widehat{\boldsymbol{\Omega}}.
\end{aligned}
$$

The most general possible expression of the form Eq. (11.224) can be written

$$\mathcal{B}_{ij}(\widehat{\mathbf{S}}, \widehat{\boldsymbol{\Omega}}) = \sum_{n=1}^{10} G^{(n)}\widehat{\mathcal{T}}_{ij}^{(n)}, \tag{11.225}$$

where the tensors $\widehat{\boldsymbol{\mathcal{T}}}^{(n)}$ are given in Table 11.6, and the coefficients can depend upon the five invariants $\widehat{S}_{ii}^2$, $\widehat{\Omega}_{ii}^2$, $\widehat{S}_{ii}^3$, $\widehat{\Omega}_{ij}^2\widehat{S}_{ji}$ and $\widehat{\Omega}_{ij}^2\widehat{S}_{ji}^2$ (Pope 1975). Like b_{ij}, each of the tensors $\widehat{\boldsymbol{\mathcal{T}}}^{(n)}$ is non-dimensional, symmetric and deviatoric. As a set they form an *integrity basis*, meaning that every symmetric deviatoric second-order tensor formed from $\widehat{\mathbf{S}}$ and $\widehat{\boldsymbol{\Omega}}$ can be expressed as a linear combination of these ten. (The proof of this is based on the Cayley-Hamilton theorem, Pope 1975.)

With the specification $G^{(1)} = -C_\mu$, $G^{(n)} = 0$ for $n > 1$, Eq. (11.225) reverts to the linear k-ε turbulent viscosity formula

$$b_{ij} = -C_\mu \widehat{S}_{ij}, \tag{11.226}$$

or, equivalently,

$$\langle u_i u_j \rangle - \tfrac{2}{3}k\delta_{ij} = -C_\mu \frac{k^2}{\varepsilon}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right). \tag{11.227}$$

A non-trivial specification of $G^{(n)}$ for $n > 1$ yields a nonlinear turbulent viscosity model, i.e., an explicit formula for $\langle u_i u_j \rangle$ that is nonlinear in the mean velocity gradients.

For flows that are statistically two-dimensional, the situation is considerably simpler. The tensors $\widehat{\boldsymbol{\mathcal{T}}}^{(1)}$, $\widehat{\boldsymbol{\mathcal{T}}}^{(2)}$ and $\widehat{\boldsymbol{\mathcal{T}}}^{(3)}$ form an integrity basis, and there are just two independent invariants $\widehat{S}^2_{kk}$ and $\widehat{\Omega}^2_{kk}$ (Pope 1975, Gatski and Speziale 1993). Consequently $G^{(4)} - G^{(10)}$ can be set to zero. Furthermore, the term in $\widehat{\boldsymbol{\mathcal{T}}}^{(3)}$ can be absorbed in the modified pressure (see Exercise 11.33) so that the value of $G^{(3)}$ has no effect on the mean velocity field. With $G^{(3)} = 0$, the nonlinear viscosity model for statistically two-dimensional flows is

$$b_{ij} = G^{(1)} \widehat{\mathcal{T}}^{(1)}_{ij} + G^{(2)} \widehat{\mathcal{T}}^{(2)}_{ij}, \qquad (11.228)$$

or, equivalently,

$$\langle u_i u_j \rangle - \tfrac{2}{3} k \delta_{ij} = 2G^{(1)} \frac{k^2}{\varepsilon} \bar{S}_{ij} + 2G^{(2)} \frac{k^3}{\varepsilon^2} (\bar{S}_{ik} \bar{\Omega}_{kj} - \bar{\Omega}_{ik} \bar{S}_{kj}). \qquad (11.229)$$

One way to obtain a suitable specification of the coefficients $G^{(n)}$ is from an algebraic stress model. Since the nonlinear turbulent viscosity formula Eq. (11.225) provides a completely general expression for b_{ij} in terms of mean velocity gradients, it follows that to every algebraic stress model, there is a corresponding nonlinear viscosity model. It is a matter of algebra to determine the corresponding coefficients $G^{(n)}$. For example, for statistically two-dimensional flows, the coefficients $G^{(n)}$ corresponding to the LRR-IP algebraic stress model are

$$G^{(1)} = -C_\mu, \quad G^{(2)} = -\lambda C_\mu, \quad G^{(3)} = 2\lambda C_\mu, \quad G^{(4-10)} = 0, \qquad (11.230)$$

where

$$\lambda \equiv \frac{(1 - C_2)}{C_R - 1 + \mathcal{P}/\varepsilon}, \qquad (11.231)$$

and

$$C_\mu \equiv \frac{\frac{2}{3}\lambda}{1 - \frac{2}{3}\lambda^2 \widehat{S}^2_{ii} - 2\lambda^2 \widehat{\Omega}^2_{ii}}, \qquad (11.232)$$

see Exercise 11.36. Figure 11.22 shows $-G^{(1)} = C_\mu$, and $-G^{(2)} = \lambda C_\mu$ as functions of $\mathcal{S}k/\varepsilon$ and $\Omega k/\varepsilon$ (where $\Omega = (2\bar{\Omega}_{ij}\bar{\Omega}_{ij})^{\frac{1}{2}}$).

(The nonlinear viscosity model defined by Eqs. (11.230)–(11.232) is not completely explicit, because the definition of λ contains $\mathcal{P}/\varepsilon = -2 b_{ij} \widehat{S}_{ij}$.

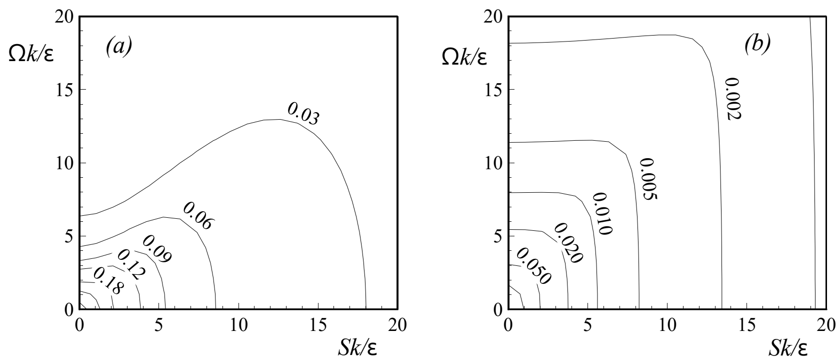

Figure 11.22: Contour plots of (a) $C_\mu = -G^{(1)}$, and (b) $-G^{(2)}$, for the LRR-IP nonlinear viscosity model (Eqs. 11.230–11.232).

Girimaji (1996) gives fully explicit formulae, obtained by solving the cubic equation for λ, see Exercise 11.36.)

Taulbee (1992) and Gatski and Speziale (1993) extend this approach to three-dimensional flows where, in general, all ten coefficients $G^{(n)}$ are non-zero.

Nonlinear viscosity models, not based on algebraic stress models, have been proposed by Yoshizawa (1984), Speziale (1987), Rubinstein and Barton (1990), Craft et al. (1996), and others. The first three mentioned are quadratic in the mean velocity gradients, and so $G^{(1)} - G^{(4)}$ are non-zero. In the model of Craft et al. $G^{(5)}$ is also non-zero. In addition to mean velocity gradients, the models of Yoshizawa and Speziale also involve $\bar{\mathrm{D}}\bar{S}_{ij}/\bar{\mathrm{D}}t$.

Exercise 11.33 Consider a statistically two-dimensional turbulent flow in the $x_1 - x_2$ plane (so that $\langle U_3 \rangle = 0$ and $\partial \langle U_i \rangle / \partial x_3 = 0$). By evaluating each component, show that

$$\widehat{S}^2_{ij} = \tfrac{1}{2}\delta^{(2)}_{ij}\widehat{S}^2_{kk}, \tag{11.233}$$

where

$$\boldsymbol{\delta}^{(2)} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11.234}$$

Hence show that

$$\widehat{\mathcal{T}}^{(3)}_{ij} = -\widehat{S}^2_{kk}\widehat{\mathcal{T}}^{(0)}_{ij}, \tag{11.235}$$

where

$$\widehat{\mathcal{T}}^{(0)}_{ij} \equiv \tfrac{1}{3}\delta_{ij} - \tfrac{1}{2}\delta^{(2)}_{ij}. \tag{11.236}$$